ICARUS **60**, 29–39 (1984)

Radial Migration of Preplanetary Material: Implications for the Accretion Time Scale Problem

KERRY HOURIGAN[1] AND WILLIAM R. WARD

Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109

Received March 22, 1984; revised June 5, 1984

Radial drift of planetesimals due to density wave interaction with the solar nebula is considered. The mechanism is most effective for large masses and provides mobility over a size range where aerodynamic drag is unimportant. The process could shorten accretion time scales to $\sim O(10^5–10^6$ years) throughout the solar system. Accumulation stalls down when growing objects are massive enough to open gaps in the gas disk. Implications of this process for current cosmogonic models are discussed.

INTRODUCTION

Of many unresolved problems regarding the formation of the planetary system, one of the more perplexing is the question of the accretion time scale. Numerous plausible mechanisms have been proposed for the rapid accumulation of material from micron-sized particles to sublunar planetesimals (see Wetherill (1980) for an excellent review). Although agreement on the specifics of these processes is by no means universal, there is a reasonable expectation that a basically correct approach could be found among those being studied. On the other hand, the final stage of accretion, i.e., the growth to planetary size, presents more difficulty.

This difficulty is partially rooted in the basic length scales that can be associated with a particle disk of surface density σ. An orbiting mass, m, has a range of strong gravitational influence of order $\lambda \sim (Gm/\Omega^2)^{1/3}$, where Ω is its orbital frequency. If one sets $m \sim \sigma\lambda^2$, i.e., the amount of disk material bounded by this range, λ becomes the basic length scale for gravitational instability, $\lambda_1 \sim G\sigma/\Omega^2$ (e.g., Safronov, 1969; Goldreich and Ward, 1973). This is the diameter of fragments of the disk that can contract due to self-gravity (provided the particle dispersion velocity is sufficiently low). The maximum mass associated with this process is $m_1 \sim G^2\sigma^3/\Omega^4$. If the mass of the debris disk, M_D, is spread out more or less uniformly, $\sigma \sim M_D/r^2$, then the number of fragments should be at least $N_1 \gtrsim M_D/m_1 \sim (M_\odot/M_D)^2$, where the substitution, $\Omega^2 \sim GM_\odot/r^3$, for a Keplerian disk, has been made. In the inner solar system, for example, $M_D \sim O(10^{28}$ g) and $N_1 \gtrsim 10^{10}$.

The differential rotation of the disk promotes further growth once breakup begins by allowing fragments of similar heliocentric distances but initially random azimuthal angles to gradually drift into proximity. Using material in an annulus of width λ_2 as the characteristic mass, $m_2 \sim \sigma r \lambda_2$, the gravitational range becomes $\lambda_2 \sim (Gm_2/\Omega^2)^{1/3} \sim (r\lambda_1)^{1/2} \sim (rG\sigma/\Omega^2)^{1/2}$. This is a second length scale associated with such a disk. For uniform σ throughout the region, strong gravitational encounters can be generated by differential motion alone (i.e., even with orbits of zero eccentricity) as long as the number of objects $N_2 \gtrsim M_D/m_2 \sim (M_\odot/M_D)^{1/2}$. In the inner solar system, $N_2 \gtrsim O(10^2)$ and $m_2 \sim O(10^{25}$ g). Considerable effort has been (and continues to be) ex-

[1] Currently at Division of Energy Technology, CSIRO, Highett, Victoria 3190, Australia.

0019-1035/84 $3.00

pended in describing aspects of particle growth throughout this size range. Not all details are agreed upon even by proponents of this accretion model, but one dominant trait characterizes these processes: strong gravitational encounters are virtually *unavoidable* and thus provide the most essential ingredient to the accumulation of material.

Once growth proceeds beyond m_2 the situation changes in a significant way. Differential semimajor axes become large enough that appreciable radial excursions must be performed to generate continued close encounters. If the average surface density remains essentially unchanged (i.e., there is no systematic radial migrations) necessary excursions must be accomplished via orbital eccentricities. Eccentricities can be produced by gravitational scattering among planetesimals, i.e., near misses. This gravitation relaxation of the disk leads to comparable orbit inclinations as well, so that the spatial density of solid material decreases as the particle disk thickens, $\rho \sim \sigma/h \sim \sigma\Omega/v$, where h is the scale height of the particle disk with dispersion velocity $v \sim O[(e,I) \times r\Omega]$. However, since the accretion rate is determined by the mass flux, ρv, this quantity is relatively insensitive to v provided it is (a) large enough to actually ensure collisions ($v > r\Omega/N$) and (b) not much smaller than a typical planetesimal's escape velocity [$v_e \sim (G\rho_p)^{1/2}(M_D/\rho_p N)^{1/3}$]. This latter condition implies that gravitational focusing is minimal. Under these conditions the growth rate of the planetesimal's radius is simply of order $dR/dt \sim \sigma\Omega/\rho_p$, where ρ_p is the object's body density. For uniform σ in the terrestrial zone, $dR/dt \sim 10\text{–}10^2$ cm/year, implying an Earth growth time of $\sim 10^7$ years. In the outer solar system surface densities of accretable matter (which include ices in this region) are not too different, but the longer orbital periods push the formation times up one to two orders of magnitude, precariously close to the total age of the solar system. These numbers are found using the minimum possible disk mass to estimate the surface density. A larger σ will obviously shorten accretion times, but if the particles are assumed uniformly distributed this will also imply mass in excess of the observed condensible constituents in the planetary system. Of course, no one expects the accretion event to be particularly efficient, but requiring the disposal of excess debris, say $10\text{–}10^2$ times the current planetary masses, seems to us rather dubious.

Aside from the long time scales, there is another fundamental complication with this model. As the planetesimal size progresses beyond the second characteristic mass, m_2, close encounters (may) cease to be inevitable. Near misses are contingent on the existence of the very orbital eccentricities they are designed to produce. This raises the question of whether such a system may be susceptible to "run-down," i.e., if eccentricities are once damped too low it may not be possible to regenerate them. This could terminate accretion at a premature stage, i.e., more numerous and smaller planets than observed (e.g., Wetherill, 1980). Indeed, it is in essence the run-down point that defines the end of this process. This point is a consequence of the relative excitation and damping mechanisms brought into play. Current modeling efforts center on devising numerical experiments in which increasingly sophisticated renditions of impacts, scattering, gas drag, etc., are incorporated in an attempt to test whether a "solar-system-like" configuration is a plausible outcome.

RADIAL DRIFT

It is clear from the preceding discussion that the surface density σ is the key parameter determining the behavior of the system. Its value could be increased locally without putting excess mass into the system only if the assumption of uniform distribution is relaxed. A nonuniform distribution could, of course, be derived from an initially uniform one by radial migration, increasing σ in some regions while decreasing it in others.

Aerodynamic Drag

A familiar mechanism capable of producing radial drift is that of aerodynamic drag (e.g., Whipple, 1972; Weidenschilling, 1977). Radial pressure gradients in the gaseous solar nebular affect its orbital velocity by slightly modifying the centripetal acceleration. As a result of drag interactions with the more slowly orbiting nebula, a particle spirals toward the Sun at a rate

$$\frac{\dot{r}}{r} \sim \frac{2\gamma\Omega(\Omega - \Omega_g)}{\gamma^2 + \Omega^2}, \tag{1}$$

where $\Omega - \Omega_g$ is the differential mean motion between the gas and a local Keplerian velocity field and $\gamma = F/mv_g$, where F is the drag force induced by gas flowing by with relative velocity, v_g. If we assume a pressure gradient of order $dP/dr \sim -c^2\rho_g/r$ where c is the sound speed, then $\Omega - \Omega_g \approx -(2\Omega\rho r)^{-1}dP/dr \approx \frac{1}{2}\Omega(c/r\Omega)^2$. The minimum decay time, which pertains to particles such that $\gamma = \Omega$, is $\tau_{\min} = (\Omega - \Omega_g)^{-1} \sim O(10^2)$ years. Such particles are $\sim O(10\text{–}10^2\text{ cm})$ in size and very mobile. If a larger object with relative orbital stability were to simply sweep up an inward flux $\sim r^2\sigma/\tau_{\min}$ of small particles, growth to planetary scale, M_p, could occur in as little as $\sim \tau_{\min}(M_p/M_D) \sim O(10^2\text{ years})$. However, as particles grow, they become increasingly decoupled from the gas. For large particles, $\gamma \ll \Omega$, the characteristic orbital decay time is $\tau = r/\dot{r} = \gamma^{-1}(r\Omega/c)^2$, i.e.,

$$\tau_{\text{drag}} \approx \frac{32}{3}(C_D\Omega)^{-1}\left(\frac{r\Omega}{c}\right)^3\left(\frac{\rho_p R}{\sigma_g}\right) \tag{2}$$

where F has been written in terms of the drag coefficient, C_D, through the relation $F = C_D\frac{1}{2}\rho_g v_g^2 \pi R^2$. At 1 AU with $c \sim 7.6 \times 10^4 T_2^{1/2}$ cm/sec, $\sigma_g \sim 10^3$ g/cm^2, $\rho_p \sim 3$ g/cm^3, and $C_D \sim 0.44$, Eq. (2) yields $\tau \approx 7 \times 10^7 R_{\text{km}} T_2^{-3/2}$ years where R_{km} is the planetesimal's radius in kilometers and the temperature of nebula is $T = T_2 \times 10^2$ °K. Since the time scale for the disk to convert to objects of size m_1 through gravitational instability is $\sim\Omega^{-1}$, particles drift only $\sim 10^{-3}$ AU in the interim. Further growth to m_2 in $\sim 10^3$ years allows only similar orbital decay, at which point the objects are effectively decoupled from the gas phase (i.e., $\tau_{\text{drag}} \gtrsim O(10^9$ years)).

If turbulence prevents the disk from clumping due to self-gravity, but accretion still proceeds by particle impacts, the growth rate is of order $\dot{R} \gtrsim O(\sigma\Omega/\rho_p) \sim 10$ cm/year. Consequently, the time interval over which the particles' size remains of order R is $\sim 10^4 R_{\text{km}}$ years, much less than that required for appreciable orbital decay. If impacts are energetic enough to prevent even individual particle growth, it is not clear what prevents virtually all material from drifting into the primordial Sun—i.e., the development of only $\leq O(10)$ accreting centers seems contrived. Again a characteristic scale comparable to planetary dimensions is not evident in this mechanism.

Tidal Drift

A mechanism that can provide mobility for objects larger than m_2 would have considerable appeal as an alternative route to planetary accumulation. Mobility seems achievable through recently studied planet–disk tidal interactions which have proved successful in explaining observed features in planetary rings (e.g., Cuzzi *et al.*, 1981; Shu *et al.*, 1983). The gravity field of a planetesimal perturbs the Keplerian flow of the gaseous disk. Perturbations become most pronounced at Lindblad resonances. Near the planetesimal the resonances become dense and their effects overlap. For many applications a reasonable approximation to their cumulative influence is provided by a continuous expres-

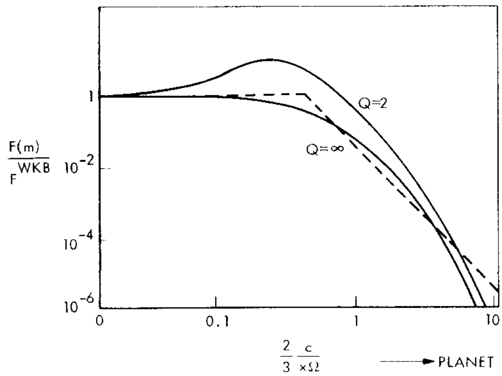

FIG. 1. Ratio of angular momentum flux from a resonance to the flux calculated from a WKB approximation. Solid curves are cases calculated by Goldreich and Tremaine (1980). Dashed curved shows the simplified torque cut-off rule used here. Q is the Toomre stability parameter, $c\Omega/\pi G\sigma$.

sion for the torque density (Lin and Papaloizou, 1979);

$$\frac{dT}{dr} \sim \frac{9f}{4} \frac{G^2 m^2 r \sigma_g}{(\Omega_p - \Omega_g)^2 x^2} \operatorname{sgn}(\Omega_p - \Omega_g). \quad (3)$$

This form breaks down at distances, x, less than about a scale height. [Inside that distance we simply assume a constant torque density equal to Eq. (3) evaluated at $x = h_*$ $\sim 1.5\ c/\Omega$. This provides the same total torque for a constant density disk as one obtains summing over Lindblad resonances. Figure 1 shows a comparison of this model with the torque density profile derived by Goldreich and Tremaine (1980). We chose $f = 2.5$ in order to match their expression at large x/h_*.] The planet experiences a net drift if outer and inner disk torques do not exactly cancel. For example, a density profile of the form $\sigma \propto r^{-k}$, integrated over the disk gives, to lowest order in (kh/r), a drift velocity of order

$$\dot{r} \sim 4fk(r\Omega)\zeta \left(\frac{\sigma_g r^2}{M_\odot}\right)\left(\frac{r}{h_*}\right)^2 \frac{m}{M_\odot} \quad (4)$$

provided the local structure of the disk is not itself substantially modified by the torque. We shall return to this point presently. [In Eq. (4), ζ is a dimensionless constant less than but of order unity that takes into account the general pressure gradient in the disk associated with the assumed density profile. This gradient shifts the positions of the Lindblad resonances in such a manner as to oppose the planetesimals' drift somewhat. Hence, Eq. (4) can be considered correct only as to its order of magnitude.] Equation (4) implies a characteristic drift time at 1 AU of order $\tau_{TD} \sim r/\dot{r} \sim 2 \times 10^{17}\ T_2\ (kf\zeta\rho_p R_{km}^3)^{-1}$ years. Note that larger masses become *more* mobile because the strength of the disk–planet torque depends on m^2. Tidal and drag time scales are comparable for objects of order $R_{km} \sim 3 \times 10^2\ T_2^{5/8} \rho_p^{-1/2} (fk\zeta)^{-1/4}$ or $m \sim 1 \times 10^{23}\ \rho_p^{-1/2} (kf\zeta)^{-3/4} T_2^{15/8}$ g. Consequently, rapidly formed sublunar-sized objects, m_2, discussed earlier, will have already bypassed the size regime dominated by drag effects.

Since growth can occur by the sweeping action of the largest and, thus, fastest objects, the characteristic growth time should be of order $\tau \sim m/\dot{m} \sim m\tau_{TD}/r^2\sigma \sim (m/M_D)\tau_{TD}$, i.e.,

$$\tau_{\text{growth}} \sim \frac{\Omega^{-1}}{4f\zeta k} \left(\frac{h_*}{r}\right)^2 \left(\frac{M_\odot}{M_D}\right)^2 \left(\frac{\sigma}{\sigma_g}\right). \quad (5)$$

For the inner solar system with $M_D \sim 10^{28}$ g, $\sigma/\sigma_g \sim 10^{-2}$, and $r \sim 1$ AU, Eq. (5) yields $\sim 2 \times 10^4\ T_2/fk\zeta$ years; for the outer solar system with $M_D \sim 3 \times 10^{29}$ g, $\sigma/\sigma_g \sim 10^{-1}$, and $r \sim 30$ AU, Eq. (5) predicts $\sim 1 \times 10^6\ T_2/fk\zeta$ years.

MASS LIMIT

The formation of both Jupiter and Saturn predated nebula loss, so that a gas-free environment must not be essential for accretion. The rapid drift and possible growth rates due to density waves indicate that such an environment may not only be unnecessary, it may actually be unlikely.

However, although the ability of large objects to migrate rapidly through the nebula potentially alleviates the time scale problem associated especially with the outer planets, it introduces another problem: without a limiting mechanism, a "first-formed" planetesimal should simply sweep the entire disk. Of course, the direction of drift depends on the sign of the density gradient, so that an object could be trapped at position of minimum gas density in the nebula. However, there is no convincing reason (presently known) to believe that the initial radial density profile of the nebula would have (several) such features. On the other hand, it is quite possible that the nebula is significantly modified by the same planet–disk torques that cause drift. Since the planetesimal "repels" the disk on either side of its orbit, this torque tends to open a gap. Once opened, the gap can stabilize the planet relative to the disk, terminating drift.

The ability of an object to open a gap depends on its mass and on conditions in the disk. We discuss two mass limits that can be associated with gap formation.

Viscous Limit

If density waves damp locally, a nonviscous disk perturbed by a stationary mass m, would evolve according to

$$\frac{D}{Dt}(\delta m l) = \delta r \frac{dT}{dr} \tag{6}$$

where $\delta m = 2\pi\sigma_g r \delta r$ is an annulus of the disk, $l = r^2\Omega$ is the specific angular momentum, and dT/dr is the torque density. The solution to Eq. (6) if dT/dr is approximated by Eq. (3) is a gap centered on the perturber that grows according to $w_g = (5t/\tau_g)^{1/5}$, where w_g is the half-width normalized to r and τ_g is a characteristic time constant

$$\tau_g = \pi(f\mu^2\Omega)^{-1}. \tag{7}$$

In deriving Eq. (7), modifications in the gas orbital frequency due to pressure gradients have been ignored as has the finite scale height of the disk. The quantity μ is the planetesimal's mass normalized to the solar mass.

In the presence of viscous shear stresses in a Keplerian disk, a couple, $g \sim 3\pi\nu\sigma r^2\Omega$, exists and one must add to the right-hand side of (6) a term of the form $-\delta r \partial g/\partial r$ to account for the ability of the gas to diffuse back into the gap. This viscosity may also lead to large-scale changes in the disk's structure and in particular to a radial flux, F. A steady-state solution for the gas profile near m is found by replacing the left-hand side of (6) with $\delta m \ \mathbf{V} \cdot \nabla l \approx \delta r \ F \ dl/dr$. Without a finite scale height, the disk–planet torque becomes an impenetrable barrier to a radial flux. It is the torque cut off associated with nonzero h that allows for leakage through the barrier. In such a case the density never goes completely to zero at $x = 0$. Neglecting changes that occur over the scale $r \gg h$, the solution for the surface density is

$$\sigma = \sigma(o)e^{3w^3|x'|/h'^4} + \mathrm{sgn}(x)\frac{h'^4F'}{3w^3}\{1 - e^{3w^3|x'|/h'^4}\}, \qquad |x'| < h'$$

$$\sigma = \sigma(o)e^{4(w/h')^3 - |w/x'|^3} - F'x'e^{-|w/x'|^3} + F' \,\mathrm{sgn}(x)\left\{e^{(w/h')^3}[1 - e^{3(w/h')^3}]\frac{h'^4}{3w^3} - (I - h')\right\}e^{-|w/x'|^3} \tag{8}$$

where $F' = F/6\pi\nu$, $h' = h_*/r$, $x' = x/r$,

$$I = \int_{h'}^{x'} [e^{(w/x')^3} - 1]dx' \tag{9}$$

and the normalized viscous gap half-width is

$$w = (r^2/9\tau_g\nu)^{1/3}. \tag{10}$$

In obtaining Eqs. (8)–(10), the viscosity is assumed constant. For zero flux this surface density reduces to

$$\begin{aligned} \sigma &= \sigma(o)e^{3w^3|x'|/h'^4} & |x'| < h' \\ &= \sigma(o)e^{4(w/h')^3 - |w/x'|^3} & |x'| > h'. \end{aligned} \tag{11}$$

If $h' > w$ the disturbance is too weak to be labeled a gap. This condition establishes a minimum mass necessary to open a gap in a nebula of viscosity ν:

$$\mu_\nu = \frac{m_\nu}{M_\odot} \geq \left[\frac{9\pi}{f}\left(\frac{\nu}{r^2\Omega}\right)\left(\frac{h}{r}\right)^3\right]^{1/2}. \tag{12}$$

[Analogous formulae have been derived for possible moonlets embedded in Saturn's rings (Lissauer *et al.*, 1981) although Voyager observations failed to detect such objects inside prominent gaps. Hence some caution may be warranted in accepting Eq. (12) without reservation.]

Inertial Limit

Equation (12) is clearly not valid at very low viscosity because radial drift of the planetesimal was neglected in its derivation, i.e., even for $\nu = 0$ there is a critical nonzero mass required for gap formation. This is simply because the time scale to open a gap as large as a scale height, $\sim h'^5\tau_g/5$, must be less than the time to drift that distance, $\sim h/\dot{r}$. Equivalently, the moment of inertia of the planetesimal must be greater than that of the local portion of the nebula with which angular momentum is being exchanged. This can be illustrated with a straightforward example. We again use Eq. (3) to represent the torque density but will ignore any shift in resonance locations due to pressure gradients. [This is a second-order effect and its omission makes the problem especially tractable. Although we believe that this approximation does not distort the qualitative behavior too severely, the reader is cautioned that this has yet to be rigorously demonstrated. We do, however, treat first-order pressure gradient effects in the Appendix and to this order of accuracy the behavior described below remains qualitatively correct.]

The equation of motion for the disk referred to a coordinate system centered on and moving with the perturber drifting with velocity, v_p, is

$$6\pi\nu \frac{\partial\sigma}{\partial x'} + 2\pi r v_p \sigma - \frac{2\pi r^2\sigma}{\tau_g x'^4}\,\mathrm{sgn}(x) = 2\pi r v_p \sigma_0 - F \quad (13)$$

where σ_0 is the unperturbed disk density, and F is the asymptotic flux observed in the stationary frame. For $\nu \to 0$, $F \to 0$ and the surface density is

$$\sigma = \sigma_0[1 - \mathrm{sgn}(x) r / v_p \tau_g x'^4]^{-1} \qquad |x'| > h'. \quad (14)$$

For $|x'| < h'$ the torque cutoff results in $\sigma(x) = \sigma(h')$. Substitution of Eq. (14) into Eq. (3) and integration over the disk replaces Eq. (4) with,

$$v_p = \frac{2\pi r^2 v_p}{m}\left(\frac{r}{v_p\tau_g}\right)^{1/4} \int \frac{\sigma_0 dz}{1 - z^4\,\mathrm{sgn}(z)} \quad (15)$$

where $z = x'(v_p\tau_g/r)^{1/4}$. Assuming $\sigma_0 \approx \sigma(0)[1 - kx' + \ldots]$, Eq. (15) can be integrated to give

$$\frac{m}{\pi\sigma h^2} = \frac{2}{H}\{G_1/h' + kG_2/H\} = \mu^*(h', k, H) \quad (16)$$

where $H^4 = v_p\tau_g h'^4/r$ and

$$G_1(H) = \frac{1}{4}\ln\left(\frac{H-1}{H+1}\right) + \frac{1}{2}\cot^{-1}(H) - \frac{2H}{H^8-1} + \frac{1}{4\sqrt{2}} \ln\left[\frac{1-\sqrt{2}\,H+H^2}{1+\sqrt{2}\,H+H^2}\right] + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}\,H}{H^2-1}\right), \quad (17)$$

$$G_2(H) = -\frac{1}{4}\ln\left(\frac{H^2-1}{H^2+1}\right) + \frac{H^6}{H^8-1} + \frac{1}{2}\cot^{-1}(H^2). \quad (18)$$

The velocity can be calculated from the relationship,

$$v_p = \frac{r}{\tau_g}\left(\frac{H}{h'}\right)^4 = \frac{f\mu^2\Omega r}{\pi}\left(\frac{H}{h'}\right)^4 = (\mu^* H^2)^2\,\pi f\left(\frac{\sigma r^2}{M_\odot}\right)^2 r\Omega. \quad (19)$$

In the limit $H \to \infty$, $\mu^* \to 0$; $\mu^* H^4 \to 4k$; and Eq. (4) (with $\zeta = 1$) is recovered. Figure 2 shows v_p/v^* as a function of μ^* where $v^* = \pi f(\sigma r^2/M_\odot)^2 r\Omega$. The drift velocity increases with mass (stable branch) up to a critical value. Past this point a stable solution does not exist, i.e., all objects will open gaps in the ($\nu = 0$) nebula. The lower curve indicates a series of unstable, lower velocity

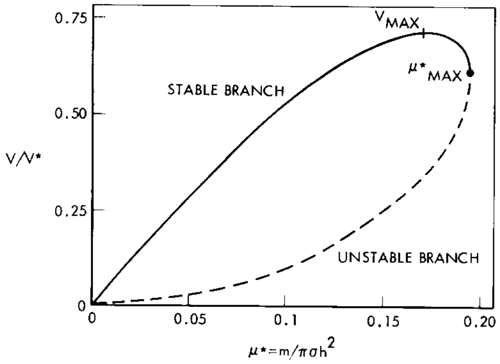

FIG. 2. Drift velocity as a function of planetesimal mass from Eqs. (16)–(19).

solutions. A slight increase in the velocity will cause acceleration until a stable v is achieved. A slight decrease will result in deceleration until drift stops and a gap opens. The planetesimal cannot escape from a gap once formed (as long as $\mu > \mu_\nu$). The inertial mass limit is thus

$$\mu_i = \mu^*_{\max} \left(\frac{\pi\sigma h^2}{M_\odot}\right) \quad (20)$$

where $\mu^*_{\max}$ is a function of k and h' that has a typical value $\sim O(10^{-1}\text{–}1)$. For example, with $k = 1.5$, $h' = 0.1$; $\mu^*_{\max} \sim 0.2$ and $m_i \sim 0.6\ \sigma h^2$. Including first-order pressure terms (see Appendix) increases these values somewhat. An improved torque density model would also change these numbers. However, it seems clear that in the inner solar system μ_i can be $\sim O(10^{27}$ g) for reasonable nebula parameters. In the outer solar system, it is about an order of magnitude or so larger.

DISCUSSION

Density waves may furnish a radial mobility for large planetesimals which relaxes the strict necessity for large eccentricities to complete the accretion process. That, together with the fact that the accretion of the giant planets was accomplished in a gaseous environment, weakens the case for gas-free accretion of the terrestrial planets (unless convincing chemical evidence to the contrary can be brought to bear). The much more rapid time scales associated with density wave assisted accretion further tighten this argument.

The density wave device is a double-edged sword, however, since without a limiting mechanism, a single massive object would eventually sweep the entire disk. In fact, even a Jovian mass would have a characteristic orbital drift time of only a few thousand years if the planet–disk interaction were not moderated in some way (Goldreich and Tremaine, 1980). Gap formation provides an effective way to terminate drift relative to the nebula and the masses associated with the inertial limit, for instance, are encouragingly close to required planetary scale for reasonable nebula parameters. However, the simplified illustration presented above assumes local damping of density waves, i.e., over distances $\leq h$. Damping mechanisms that have been suggested as effective in particle disks, such as viscous damping and/or shock development in nonlinear waves (Goldreich and Tremaine, 1980) are less certain in a low mass gaseous nebula with a large Toomre stability number, $Q = c^2\Omega/\pi G\sigma \gg 1$ (Ward, 1984). Undamped waves carry angular momentum to more remote regions of the disk before deposition. This process retards gap formation so that Eq. (20) is only a lower limit to the critical mass. Clearly a more realistic model of local planet–disk interactions, including both pressure gradient effects and detailed treatments of various wave damping mechanisms must be performed before the behavior of such a system can be described with any reliability. Although such work is in progress, we are content here to provide only an order-of-magnitude assessment of these issues.

We close this discussion by raising one more point of interest. Even if a convincing argument for the termination of radial drift at or near planetary size can eventually be mounted, the nebula is still in place and its removal seems to pose some serious problems. Viscous shear stresses associated with the Keplerian motion of the disk are currently regarded as a likely method of disk dispersal (e.g., Cassen and Moosman, 1982). Outward angular momentum transport results in a spreading of the more remote portions of the disk while most of the gas flows to the center and is presumably added to the solar mass (Lynden-Bell and Pringle, 1974). This process is assumed to occur contemporaneously with planetary accretion. This is in contrast to "preaccretion disk theory" models which often evoked a dispersal event following a relatively quiescent period for the nebula during which the planets, at least the giant ones, accumulated. [The most commonly proposed trigger for this ejection phase has been a T-tauri-like stage of solar evolution.] Its apparent freedom from such ad hoc assumptions is a compelling feature of viscous accretion disk models of the solar nebula. However, closer inspection reveals that this method of nebula dissipation may have its own difficulties, particularly when operating in conjunction with density waves.

For shear stresses to be at all important in a global sense the nebula must be turbulent to some degree because molecular viscosity is too small to be a significant factor by several orders of magnitude. The characteristic time scale for viscous evolution of the nebula is $\tau_N \sim r^2/\nu$, where ν denotes an effective turbulent viscosity. The degree of turbulence and, indeed, whether such turbulence can persist for a time τ_N, is an active subject of research. Several mechanisms for generating turbulence have been suggested (Cameron, 1978; Lin and Papaloizou, 1980). The most convincing case to date is that made by Lin and Papaloizou who argue that grain opacity keeps the nebula unstable to convective overturn in the vertical direction and that the resulting gas eddies also couple radial motions over distances comparable to a scale height. Some uncertainty exists, however, as to the strength and persistence of this process over times of order τ_N, since turbulence promotes grain coagulation which, in turn, drops the opacity (Weidenschilling, 1983). Nevertheless, examination of the ramifications of various *assumed* values for τ_N leads to the some interesting observations:

(1) Mild turbulence such that $\mu_\nu < \mu_i$ may allow the inertial limit to operate and prevent a planet's overgrowth by an unchecked sweep of the disk. However, the implied nebula evolution time, $\tau_N > O[\Omega^{-1}\mu_\nu^{-2}(h/r)^3] \sim O(10^9 \text{ years})$, will be too long to account for disk dispersal. Furthermore, as mentioned above, a case for local wave damping in a low viscosity disc must be made.

(2) Disk evolution with $\tau_N \sim O(10^5 \text{ years})$ could, of course, simply suspend radial drift by removing the gas phase; but the timing would seem overly fortuitous to still allow the accretion process to proceed to the point of only $O(10)$ objects.

(3) The accumulation of Jupiter and Saturn must predate, or at least be contemporaneous with, disk removal. These objects exceed μ_ν if $\tau_N \sim O(10^5 \text{ years})$ and will open gaps, preventing further motion relative to disk material. Continued viscous evolution of the nebula will force these objects to suffer orbital decay (Ward, 1982; Hourigan and Ward, 1983). If uninterrupted by some intervening event, planetary material so locked into the momentum transport process of the gas disk may eventually drift into the primordial Sun.

(4) If $\tau_N < O(10^4 \text{ years})$, viscous diffusion will suppress gap formation even for the largest planets. However, disk–planet interactions are so powerful for a Jovian-sized object that the characteristic orbital drift time is only $O(10^4 \text{ years})$ (Goldreich and Tremaine, 1980), again putting the stability of the newly formed planetary system in considerable doubt. As an additional complication, one must account for the failure of Uranus and Neptune to accrete substantial amounts of hydrogen and helium during the nebula's lifetime.

(5) Finally, although extremely vigorous

turbulence, i.e., $\tau_N \sim O(10^2$–10^3 years) could remove the disk before severe changes in the giant planet orbits could occur, such a short disk lifetime is clearly an inadequate interval for the accretion of planet-sized objects. Although one could rely on gas-free accretion to account for the terrestrial planets, we are left with no way to explain the accumulation of the giant planet cores.

The above considerations indicate some of the difficulties in constructing a simple nebula model that satisfies all required constraints. Although both density waves and viscous shear stresses are important and powerful processes that may (among other things) help explain planetary accretion and nebula dispersal, their incorporation into models of solar system origin is not without challenge.

APPENDIX

First-Order Treatment of Pressure Gradient

Equation (6) can be written,

$$2\pi r\sigma \left\{\frac{\partial}{\partial t}(r^2\Omega) + v\frac{\partial}{\partial t}(r^2\Omega)\right\} = \frac{dT}{dr}. \quad \text{(A1)}$$

Denoting the radial flux $F = 2\pi r\sigma v$ and using $\partial r/\partial t = 0$ yield

$$2\pi r^3\sigma\frac{\partial\Omega}{\partial t} + F\frac{\partial(r^2\Omega)}{\partial r} = \frac{dT}{dr}. \quad \text{(A2)}$$

From the continuity equation,

$$\frac{\partial\sigma}{\partial t} = -\frac{1}{2\pi r}\frac{\partial F}{\partial r}, \quad \text{(A3)}$$

whereas the gas orbital frequency, Ω, taking into account the radial pressure gradient is

$$\Omega^2 \approx \Omega_K^2 + \frac{c^2}{r\sigma}\frac{\partial\sigma}{\partial r}. \quad \text{(A4)}$$

In (A4), $\Omega_K^2 = GM_\odot/r^3$ is the Keplerian frequency; the horizontal pressure is assumed to satisfy the polytropic equation of state, $p = K\sigma^\gamma$; and $c^2 = \gamma P/\sigma$ is the gas sound speed. Differentiating (A4) with respect to time, and substituting (A3) for $\partial\sigma/\partial t$ yields

$$2\Omega\frac{\partial\Omega}{\partial t} = \frac{c^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{\sigma}\frac{\partial\sigma}{\partial t}\right) = -\frac{c^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{2\pi r\sigma}\frac{\partial F}{\partial r}\right). \quad \text{(A5)}$$

In deriving (A5), variations in c^2 have been ignored. Incorporation of (A5) into (A1) then leads to

$$-\frac{r^2\sigma c^2}{2\Omega}\frac{\partial}{\partial r}\left(\frac{1}{r\sigma}\frac{\partial F}{\partial r}\right) + F\frac{\partial}{\partial r}(r^2\Omega) = \frac{dT}{dr}. \quad \text{(A6)}$$

Finally, differentiating (A4) with respect to r,

$$\frac{\partial\Omega^2}{\partial r} = -\frac{3}{2}\frac{\Omega_K^2}{r} + c^2\frac{\partial}{\partial r}\left(\frac{1}{r\sigma}\frac{\partial\sigma}{\partial r}\right) \quad \text{(A7)}$$

and combining with (A6), the radial flux equation reads,

$$-\frac{1}{2}\sigma r^2c^2\frac{\partial}{\partial r}\left(\frac{1}{r\sigma}\frac{\partial F}{\partial r}\right) + F\left\{\frac{1}{2}r\Omega_K^2 + \frac{1}{2}\left(\frac{c}{r}\right)^2\frac{\partial}{\partial r}\left(\frac{r^3}{\sigma}\frac{\partial\sigma}{\partial r}\right)\right\} = \Omega\frac{dT}{dr}. \quad \text{(A8)}$$

The radial flux F is assumed first order in the disturbing torque. A first-order solution is found by substituting the undisturbed nebula surface density $\sigma_0(r)$ for σ everywhere on the left-hand side to give, after some rearranging,

$$\frac{\partial^2F}{\partial r^2} - \left[\frac{\partial}{\partial r}\ln(r\sigma_0)\right]\frac{\partial F}{\partial r} - \left\{\left(\frac{\Omega_K}{c}\right)^2 + \frac{1}{r^3}\frac{\partial}{\partial r}\left(\frac{r^3}{\sigma_0}\frac{\partial\sigma_0}{\partial r}\right)\right\}F = -\frac{2\Omega}{rc^2}\frac{dT}{dr}. \quad \text{(A9)}$$

Equation (3) is used for dT/dr. To this order of accuracy, undisturbed values are substituted for all terms on the right. [Note, that shifts in the positions of the various Lindblad resources are thus not included in first-order treatment.] Finally, if we ignore slowly changing terms and retain only portions of (A9) that vary most strongly with $x = r - r_p$, the problem simplifies to

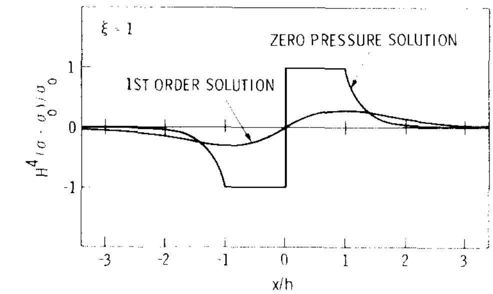

FIG. 3. Comparison of calculated nebula disturbances for zero-order and first-order pressure solutions.

$$\frac{\partial^2 F}{\partial x^2} - \left(\frac{\Omega_K}{c}\right)^2 F = -\frac{9fG^2M^2\Omega\sigma_0}{2c^2(\Omega_p - \Omega)^2x^2}\,\mathrm{sgn}(\Omega_p - \Omega). \quad \text{(A10)}$$

Again, the right-hand torque density is valid only for x larger than about a scale height. The torque "cut-off" zone is more or less centered on the corotation point which lies at $x_{c.r.} \approx \frac{1}{3}(c/\Omega_p)^2\sigma_0^{-1}\partial\sigma_0/\partial r$. Comparison of this first-order treatment with the zero pressure solution employed in the text is accomplished most easily by considering a σ_0 = constant nebula as an example. In this case the right-hand torque reduced to

$$\frac{-2\pi r^6\sigma_0}{x^4\tau_g}\left(\frac{\Omega_K}{c}\right)^2 \mathrm{sgn}(x) \qquad |x| > h_*. \quad \text{(A11)}$$

The solution of (A10) and (A11) can be combined with Eq. (A3) to find

$$\sigma - \sigma_0 = -\frac{1}{2\pi}\int \frac{1}{r}\frac{\partial F}{\partial r}\,dt \approx -\frac{1}{2\pi r}\int \frac{\partial F}{\partial x}\,dt = \frac{1}{2\pi r v_p}\int \frac{\partial F}{\partial t}\,dt = F/2\pi r v_p. \quad \text{(A12)}$$

Where we have used $\dot{r}_p = v_p$. In the limit $c \to 0$, (A10)–(A12) reduces to the first-order version of Eq. (14), i.e.,

$$\sigma - \sigma_0 \approx \frac{\sigma_0}{x'^4}\left(\frac{r}{v_p t_g}\right)\mathrm{sgn}(x) \quad \text{(A13)}$$

(except inside $|x'| < h'$ where x' is replaced by h'). For $c \neq 0$, the solution to (A10)–(A12) is

$$\sigma - \sigma_0 \approx \frac{\sigma_0}{h'^4}\left(\frac{r}{v_p t_g}\right) \cdot \begin{cases} \dfrac{1}{\xi^3}E_4(\xi)\sinh(x/h) + \dfrac{1}{2}\dfrac{(1-e^{-\xi})^2}{\xi^4}e^{\xi - x/h} + \displaystyle\int_\xi^{x/h}\sinh(t - x/h)t^{-4}dt, & x > h_* \\ \dfrac{E_4(\xi)}{\xi^3} - \dfrac{e^{-\xi}}{\xi^4}\sinh(x/h) + \dfrac{1}{\xi^4}(1 - e^{-x/h}), & x < h_* \end{cases} \quad \text{(A14)}$$

together with $\sigma(-x) = -\sigma(x)$ and, $\xi = h_*/h$. The integral in (A14) can in turn be expressed in terms of exponential integrals as

$$\frac{1}{12}e^{-x/h}\left[Ei\left(\frac{x}{h}\right) - Ei(\xi)\right] - \frac{1}{2}e^{x/h}\left[\xi^{-3}E_4(\xi) - \left(\frac{h}{x}\right)^3 E_4\left(\frac{x}{h}\right)\right] + \frac{1}{12}\frac{e^{\xi - x/h}}{\xi^3}(2 + \xi + \xi^2) - \frac{1}{12}\left(\frac{h}{x}\right)^3\left(2 + \frac{x}{h} + 2\left(\frac{x}{h}\right)^2\right) \quad \text{(A15)}$$

where the exponential integrals,

$$Ei(x) = \int_{-\infty}^{x}\frac{e^t}{t}\,dt$$
$$E_n(x) = \int_1^{\infty}\frac{e^{-xt}}{t^n}\,dt, \quad \text{(A16)}$$

are well tabulated functions. Figure 3 compares solution (A13) and (A14) for the case $\xi = 1$. The artificial discontinuity at $x = 0$ is removed and the disturbance is weaker by a factor ~3, but the general qualitative features that led to the prediction of an inertial limit remain the same.

ACKNOWLEDGMENTS

This work was supported at JPL in part by Contract NAS7-100 with the National Aeronautics and Space Administration. In addition, K.H. held a Resident Research Associateship from the National Research Council while at JPL.

REFERENCES

CAMERON, A. G. W. (1978). Physics of the primitive solar accretion disc. *Moon Planets* **8,** 5–40.

CASSEN, P., AND M. MOOSMAN (1982). On the formation of protostellar disks. *Icarus* **48,** 353–376.

CUZZI, J. N., J. J. LISSAUER, AND F. H. SHU (1981). Density waves in Saturn's rings. *Nature* **292,** 703–707.

GOLDREICH, P., AND S. TRAMAINE (1980). Disk–satellite interactions. *Astrophys. J.* **241,** 425–441.

GOLDREICH, P., AND W. R. WARD (1973). The formation of planetesimals. *Astrophys. J.* **183,** 1051.

HOURIGAN, K., AND W. R. WARD (1983). Planetary aggregation and nebula tides. *Lunar Planet. Sci. XIV.*

LIN, D. N. C., AND J. PAPALOIZOU (1979). Tidal torques on accretion discs in binary systems with extreme mass ratio. *Mon. Not. R. Astron. Soc.* **186,** 799–812.

LIN, D. N. C., AND J. PAPALOIZOU (1980). On the structure and evolution of the primordial solar nebula. *Mon. Not. R. Astr. Soc.* **191,** 37–48.

LISSAUER, J. J., F. H. SHU, AND J. N. CUZZI (1981). Moonlets in Saturn's ring? *Nature* **292,** 707–711.

LYNDEN-BELL, D., AND J. E. PRINGLE (1974). The evolution of viscous discs and the origin of the nebula variables. *Mon. Not. R. Astron. Soc.* **168,** 603–637.

SAFRONOV, V. S. (1969). *Evolution of the Protoplanetary Cloud and the Formation of the Earth and Planets* (Transl. Israel Program for Scientific Translation (1972), NASA TTF-677). Nauka, Moscow.

SHU, F. H., J. N. CUZZI, AND J. J. LISSAUER (1983). Bending waves in Saturn's Rings. *Icarus* **53,** 185–207.

WARD, W. R. (1982). Tidal barriers in the solar nebula. *Lunar Planetary Sci. XIII,* 831.

WARD, W. R. (1984). The solar nebula and the planetesimal disc. In *Planetary Rings* (Brahic and Greenberg, Eds.), 660–684.

WEIDENSCHILLING, S. J. (1977). Aerodynamics of solid bodies in the solar nebula. *Mon. Not. R. Astron. Soc.* **180,** 57–70.

WEIDENSCHILLING, S. J. (1983). Dust in a turbulent solar nebula. *Bull. Amer. Astron. Soc.* (abstract), 811.

WETHERILL, G. W. (1980). Formation of the terrestrial planets. *Annu. Rev. Astron. Astrophys.* **18,** 77–113.

WHIPPLE, F. (1972). On certain aerodynamic processes for asteroids and comets. In *From Plasma to Planet* (A. Elvins, Ed.), pp. 211–232. Wiley, New York.